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On some topological invariants of algebraic functions associated to the Young stratification of polynomials

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Abstract

The connections between braid groups, algebraic functions and spaces of generic polynomials have been described by Arnol'd in his famous paper on topological invariants of algebraic functions. In this seminal paper Arnol'd studied the cohomology rings of the spaces of polynomials without multiple roots and showed that these rings satisfy three theorems: the stabilization theorem (the rings stabilize has the degree of the polynomials tends to infinity), the repetition theorem (the rings associated to spaces of polynomials of successive even and odd degrees are isomorphic) and the finiteness theorem (all cohomology groups are finite except the first two). We generalize the results of Arnol'd by considering cohomology rings dual to arbitrary Young strata of the space of polynomials. We show that the stabilization, repetition and finiteness theorems still hold *mutatis mutandis* in this more general context.

Introduction

A generic complex polynomial of degree n of the form $z^n + \lambda_1 z^{n-1} + \cdots + \lambda_n$ has n distinct roots. In the space of polynomials of degree n the subspace of polynomials with multiple roots form a hypersurface called the *discriminant*. In [1] Arnol'd described many interesting connections between braid groups, algebraic functions and spaces of generic polynomials. For instance the space $P^n(2)$ of complex polynomials of degree n without

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multiple roots is an Eilenberg MacLane space $K(\pi, 1)$ for the group $\mathrm{Br}(n)$ of braids with n strings:

$$\pi_1\big(P^n(2)\big) = \mathrm{Br}(n), \qquad \pi_i\big(P^n(2)\big) = 0 \quad \text{for } i > 1.$$

Moreover $P^n(2)$ is the complement of the branching manifold of the universal n-valued algebraic function $z(\lambda_1, \dots, \lambda_n)$ defined by the algebraic equation:

$$z^n + \lambda_1 z^{n-1} + \cdots + \lambda_n = 0$$

(any n-valued algebraic function gives rise to a polynomial mapping between the space of its arguments and the space of arguments $\{\lambda\}$ of this universal function). According to these connections, the cohomology rings of $P^n(2)$ and $\mathrm{Br}(n)$ coincide and the cohomology classes of $P^n(2)$ give rise to characteristic classes of algebraic functions. Using these characteristic classes, Arnol'd proved in [2] that algebraic functions of a certain number of variables can not be represented by complete superposition of algebraic functions of fewer variables (complete superposition means that the original algebraic function coincide exactly with the superposition and not only with one of its irreducible branch). The results of Arnol'd on superposition of algebraic functions were later improved by Lin using other methods [4,5].

The main results of [1] concerned the cohomology rings of the spaces of degree n polynomials without k roots of multiplicity at least m. Here we generalize the theorems of finiteness, repetition and stabilization proved by Arnol'd to spaces of polynomials whose multiple roots do not obey any prescribe pattern of multiplicities and not only repetition patterns of the form $(m, \dots, m)$ (some of these results have been announced in [7]). Special cases of this problem were considered by Sundaram and Welker in [8]. In particular, our finiteness theorem implies conjecture 4.12. of [8].

The Young stratification of the space of polynomials is described in section one. The main theorems are enunciated in section two and proved in section five. The proofs are based on resolutions of self-intersections of the strata (section three) and the construction of a natural embedding of the space of polynomials of a given degree into the space of polynomials of degree one more (section four). The results of explicit computations of cohomology groups in lowest dimensions are given in section six.

1. Young strata of the discriminant

Consider a unitary polynomial of degree n: $z^n + \lambda_1 z^{n-1} + \cdots + \lambda_n$ (in the following we denote by P^n the space of unitary polynomials of degree n). If this polynomial belongs to the *discriminant* we can consider the collection of multiplicities of its multiple roots. This collection is called the *Young diagram* of the polynomial.

Example 1. A generic polynomial of degree 4 has four distinct roots. A generic polynomial in the discriminant has exactly one root of multiplicity two (its Young diagram is (2)). More degenerate polynomials have either two double roots (Young diagram $(2, 2)$), a triple root (Young diagram (3)) or a root of multiplicity four (Young diagram (4)).

$$\begin{tikzcd}
\Sigma(2) & \Sigma(3) \arrow[l] & \Sigma(4) \arrow[l] \arrow[dl] \\
& \Sigma(2,2) \arrow[ul] & \Sigma(3,2) \arrow[l] \arrow[ul] \\
& & \Sigma(2,2,2) \arrow[ul]
\end{tikzcd}$$

Fig. 1. Strata of low codimension and their adjacencies (only the multiplicities of multiple roots are indicated).

Given a collection of integers in descending order $(m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k)$, $m_k \geqslant 2$, we denote by $\Sigma^n(m_1, m_2, \ldots, m_k)$ the space of unitary polynomials of degree n having roots of multiplicities at least $m_1, m_2, \ldots, m_k$; this is the space of polynomials which can be factorized in the following way:

$$p(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k} q(z), \tag{1}$$

where $z_1, z_2, \ldots, z_k$ are, not necessarily distinct, complex numbers and q is a unitary polynomial of degree $n - (m_1 + m_2 + \cdots + m_k)$.

Remark 2. According to this definition the strata $\Sigma^n(m_1, \ldots, m_k)$ are closed algebraic subset of P^n.

Remark 3. For $Y = (m_1, \ldots, m_k)$ and $n > \sum_{i=1}^k m_k$ the real codimension of the stratum $\Sigma^n Y$ in P^n is $2\sum_{i=1}^k (m_k - 1)$. In particular each multiplicity m_i in the Young diagram $(m_1, \ldots, m_k)$ gives a contribution of $2m_i - 2$ to the codimension of the stratum.

Remark 4. If n is large enough the stratum $\Sigma^n(m_1, \ldots, m_k)$ is singular. In particular it has self-intersections (see Section 3).

The strata of low codimension are indicated in Fig. 1 together with their adjacencies (in this figure we have not indicated the degree of polynomials but only their singularities).

Remark 5. Their exists a profound connection between the first line of this diagram (the spaces of polynomials with a root of high multiplicity) and the main diagonal (the spaces of polynomials with many double roots). This relation is investigated in [3,6].

Given a Young diagram $(m_1, m_2, \ldots, m_k)$ denote by $P^n(m_1, m_2, \ldots, m_k)$ the complement of the stratum $\Sigma^n(m_1, m_2, \ldots, m_k)$ in P^n. This is the space of polynomials which can not be factorized as in Eq. (1). For instance $P^n(2, 2)$ is the space of degree n polynomials having at most one double root or one root of multiplicity three.

2. Finiteness, repetition, stabilization

Consider a Young diagram $Y = (m_1, m_2, \ldots, m_k)$. Denote by $\operatorname{codim} Y$ the "formal codimension" of Y: $\operatorname{codim} Y = \sum_{i=0}^k 2(m_i - 1)$ (see Remark 3). Denote by $\min Y = m_k$

the minimum of the collection Y and by $\gcd Y$ the greatest common divisor of the integers $m_1, m_2, \ldots, m_k$ occurring in Y.

Theorem 6 (Finiteness). *The cohomology groups of the space $P^n Y$ are all finite except $H^0(P^n Y) = \mathbb{Z}$ and $H^{\operatorname{codim} Y - 1}(P^n Y) = \mathbb{Z}$.*

Consider the sequence of continuous maps $\Phi^n : P^n \to P^{n+1}$ defined as follows: the map Φ^n associates to the polynomial with roots $z_1, z_2, \ldots, z_n$ the polynomial with roots $z_1, z_2, \ldots, z_n$ and $\sum_{i=0}^{n}(|z_i| + 1)$, where $|z_i|$ denotes the modulus of z_i. These maps preserve the multiplicities of roots and hence the strata $\Sigma(m_1, m_2, \ldots, m_k)$. In particular for any Young diagram $Y = (m_1, \ldots, m_k)$ we have a sequence of embeddings:

$$P^1 Y \to P^2 Y \to P^3 Y \to \cdots.$$

Theorem 7 (Repetition). *For any integers q and r such that $0 \leqslant r < \gcd Y$ the cohomology groups of the spaces $P^{q \gcd Y}$ and $P^{q \gcd Y + r}$ are all equal. Moreover the isomorphisms between these groups are induced by the maps Φ.*

Theorem 8 (Stabilization). *For any integer n, the embedding Φ^n induces isomorphisms of cohomology groups*:

$$H^i\bigl(P^n Y\bigr) \approx H^i\bigl(P^{n+1} Y\bigr) \quad \textit{for } i \leqslant \frac{(n - \operatorname{sum} Y)(2 \min Y - 3)}{\min Y} + \operatorname{codim} Y - 1.$$

Explicit results for the first cohomology groups of strata of low codimension are given in Section 6.

Denote by $\overline{\Sigma}^n Y$ the one point compactification of the stratum $\Sigma^n Y$. The cohomology groups of $P^n Y$ and the homology groups of $\overline{\Sigma}^n Y$ are related by Alexander duality:

$$H^i\bigl(P^n Y\bigr) = \widetilde{H}_{2n-i-1}\bigl(\overline{\Sigma}^n Y\bigr) \quad \text{for } i > 0,$$

where $\widetilde{H}$ denotes the reduced homology modulo the point at infinity. Hence it suffices to prove the duals of Theorems 6 to 8. To simplify we denote in the following by $H_{(i)}(\overline{\Sigma}^n Y) = \widetilde{H}_{2n-i-1}(\overline{\Sigma}^n Y)$ the *transition homology*.

Remark 9. In the case $Y = (m, \ldots, m)$ Theorems 6, 7 and 8 imply the finiteness, repetition and stabilization theorems of [1].

3. Resolution of singularities

Consider a stratum $\Sigma^n(m_1, m_2, \ldots, m_k)$. If n is large enough this stratum contains self-intersections: the self-intersection of this stratum is the union of all strata $\Sigma^n(m_1', m_2', \ldots, m_l')$ such that $m_1', m_2', \ldots, m_l'$ is obtained from $m_1, m_2, \ldots, m_k$ by repeating some of the m_i. The proofs of Theorems 6 to 8 are based on some special “resolutions” of these self-intersections.

Remark 10. A polynomial belongs to the self-intersection of $\Sigma^n Y$ if and only if it can be factorized as shown in equation 1 in more than one way.

The resolution of a stratum self-intersections is done in the following way. Consider a Young diagram $Y = (m_1, \dots, m_1, \dots, m_k, \dots, m_k)$, where each m_i appears n_i times and all the m_i corresponding to different i are different: $m_1 > m_2 > \cdots > m_k$. A polynomial in the stratum $\Sigma^n Y$ can be factorized as follows:

$$p(z) = q(z) \times \prod_{i=1}^{k} \prod_{j=1}^{n_k} (z - z_{i,j})^{m_i}, \tag{2}$$

where $q(z)$ is a unitary polynomial of degree $n - \sum_{i=1}^{k} n_i m_i$ and the $z_{i,j}$ are arbitrary complex numbers.

Denote by Y' the Young diagram obtained from Y by removing all the m_k. Consider the map π:

$$\begin{aligned} \pi : P^{n_k} \times \Sigma^{n-n_k m_k}(Y') &\to \Sigma^n Y, \\ \big(p(z), q(z)\big) &\mapsto p(z)^{m_k} \times q(z). \end{aligned} \tag{3}$$

This map is continuous, onto and of degree one (the preimage of a generic point of $\Sigma^n Y$ under this map consists of exactly one point).

Given a topological space X, with a marked point ∞, denote by $\overline{\mathbb{C}}^k \sharp X$ the $2k$-fold suspension of X (we consider the suspension of pointed spaces as defined in [1]).

Remark 11. The space P^{n_k} is isomorphic to $\mathbb{C}^{n_k}$. In particular the one point compactification of P^{n_k} is a sphere of real dimension $2n_k$. Hence according to our definition of the transition homology, for any integer k, the homology groups of $\overline{P}^k \sharp \overline{X}$ and $\overline{X}$ are isomorphic.

Denote by A the space $\overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y'')$, where Y'' is the Young diagram obtained from Y' by adding one m_k. Denote by B the space $\overline{\Sigma}(Y''')$, where Y''' is the Young diagram obtained from Y by adding one m_k (the space $\Sigma(Y''')$ is an irreducible component of the self-intersection stratum of ΣY). The map π gives rise to the following commutative diagram, where the horizontal arrows are inclusions and the vertical arrows are induced by π:

$$\begin{tikzcd} \overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y') \arrow[d, "\pi"'] & \overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y'') \arrow[l] \arrow[d, "\pi"] \\ \overline{\Sigma}^n Y & \overline{\Sigma}^n(Y''') \arrow[l] \end{tikzcd} \tag{4}$$

Lemma 12. *The commutative diagram* (4) *induces an homeomorphism*:

$$\overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y') \setminus \overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y'') \approx \overline{\Sigma}^n Y \setminus \overline{\Sigma}^n(Y''').$$

Proof. By construction any polynomial in $\Sigma^n Y \setminus \Sigma^n(Y''')$ has exactly one preimage by π. Moreover since the multiplicities m_i for different i are distinct it is easy to see that the restriction of π from the complement of $\overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}^{n-n_k m_k}(Y'')$ to the complement of $\overline{\Sigma}^n(Y''')$ is continuously invertible. $\square$

The same process can be repeated for $\overline{\Sigma}(Y')$ and so on. In the end, after k resolutions, we obtain a commutative diagram of the following form (the vertical arrows are induced by π and the horizontal arrows are embeddings):

$$\begin{tikzcd} S_k \arrow[d, "\pi"'] & A_k \arrow[l] \arrow[d, "\pi"] \\ \overline{\Sigma}^n Y & B_k \arrow[l] \end{tikzcd} \tag{5}$$

where $S_k = \overline{\mathbb{C}}^{n_1} \sharp \cdots \sharp \overline{\mathbb{C}}^{n_k} \sharp \overline{P}^{n-(n_k m_k + \cdots + n_1 m_1)}$ is a sphere, B_k is the self-intersection stratum of $\overline{\Sigma} Y$ and A_k is the preimage of B_k. This diagram corresponds to the factorization given in Eq. (2) of polynomials belonging to $\overline{\Sigma} Y$. It induces a homeomorphism $S_k \setminus A_k \approx \overline{\Sigma}^n Y \setminus B_k$. In particular every stratum $\overline{\Sigma} Y$ is obtained from a sphere by a mapping of degree 1.

Example 13. Consider the stratum $\overline{\Sigma}^n(3,3,2)$. The resolution of this stratum consists of the following diagram:

$$\begin{tikzcd} \overline{\mathbb{C}}^2 \sharp \overline{\mathbb{C}} \sharp P^{n-8} \arrow[d] & \overline{\mathbb{C}}^2 \sharp \overline{\mathbb{C}} \sharp (\overline{\Sigma}^{n-8}(3) \cup \overline{\Sigma}^{n-8}(2)) \arrow[l] \arrow[d] \\ \overline{\Sigma}^n(3,3,2) & \overline{\Sigma}^n(3,3,3,2) \cup \overline{\Sigma}^n(3,3,2,2) \arrow[l] \end{tikzcd}$$

Given a Young diagram Y denote by $\operatorname{sum} Y$ the sum of all integers occurring in Y counted with repetitions. The integer $\operatorname{sum} Y$ is the minimal degree of a polynomial having multiple roots whose multiplicities satisfy the pattern given by Y. There are some special cases in which the resolution of the stratum ΣY is trivial. These are given by the following lemma:

Lemma 14. *If* $\operatorname{sum} Y \leqslant n < \operatorname{sum} Y + \min Y$ *then the set* A_k *is void. In particular the stratum* $\overline{\Sigma} Y$ *is homeomorphic to the sphere* S_k.

Proof. Under the condition of the lemma, any polynomial belonging to $\overline{\Sigma}^n Y$ can be factorized as in Eq. (2) in only one way. In particular the self-intersection stratum B_k of $\overline{\Sigma} Y$ is void and so is A_k. $\square$

Denote by S the space $\overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}(Y')$, by A the space $\overline{\mathbb{C}}^{n_k} \sharp \overline{\Sigma}(Y'')$ and by B the space $\overline{\Sigma}(Y''')$ obtained after the first resolution. According to Remark 11 the following isomorphisms hold:

$$\begin{aligned} &H_{(i)}(S) \approx H_{(i)}\big(\overline{\Sigma}(Y')\big), \qquad H_{(i)}(A) \approx H_{(i)}\big(\overline{\Sigma}(Y'')\big), \\ &H_{(i)}(B) \approx H_{(i)}\big(\overline{\Sigma}(Y''')\big). \end{aligned} \tag{6}$$

Let μ be the codimension of A in S: $\mu = 2\min Y - 2$. From the exact sequences of the pairs (S, A) and $(\overline{\Sigma}^n Y, B)$ we can form, for $i > 0$, a commutative diagram of exact sequences (the vertical arrows are induced by π):

$$\begin{tikzcd}
H_{(i-1)}(A) \arrow[r] \arrow[d] & H_{(i+\mu-1)}(S) \arrow[r] \arrow[d] & H_{(i+\mu-1)}(S/A) \arrow[r] \arrow[d] & H_{(i)}(A) \arrow[d] \\
H_{(i-1)}(B) \arrow[r] & H_{(i+\mu-1)}(\overline{\Sigma}^n Y) \arrow[r] & H_{(i+\mu-1)}(\overline{\Sigma}^n Y/B) \arrow[r] & H_{(i)}(B)
\end{tikzcd} \tag{7}$$

By Lemma 12 the vertical arrow in the middle of this diagram: $H_{(i+\mu-1)}(S/A) \to H_{(i+\mu-1)}(\overline{\Sigma}^n Y/B)$ is an isomorphism.

4. Degree shift

To prove Theorems 6 to 8 we need to relate explicitly the homology groups of the strata $\overline{\Sigma}^n Y$ with different n. This is done by considering the neighborhood of the points z^n and $z^{n-1}(z-1)$ in P^n.

Lemma 15. *Any stratum $\Sigma^n Y$ is a quasi-homogeneous subspace of P^n.*

Proof. For any $\lambda \in \mathbb{R} \setminus \{0\}$ define $D_\lambda : P^n \to P^n$ as the map sending the polynomial with roots $z_1, \ldots, z_n$ to the polynomial with roots $\lambda z_1, \ldots, \lambda z_n$. Using the coordinates $a_1, \ldots, a_n$ on the space $P^n = \{z^n + a_1 z^{n-1} + \cdots + a_n\}$, the map D_λ is given by

$$D_\lambda : (a_1, \ldots, a_n) \mapsto \big(\lambda a_1, \ldots, \lambda^n a_n\big).$$

Since any stratum $\Sigma^n Y$ is globally invariant by the maps D_λ, it is quasi-homogeneous in the variables $a_1, \ldots, a_n$ with the weights $1, \ldots, n$. $\square$

Define $V^n(\varepsilon)$ as the subspace of P^n of polynomials whose roots have modulus less than ε. Lemma 15 implies:

Corollary 16. *The space $\Sigma^n Y$ is homeomorphic to its intersection with $V^n(\varepsilon)$ for any* $\varepsilon > 0$.

Define $U^n(\varepsilon)$ as the subspace of P^n of polynomials having $n-1$ roots with modulus less than ε and one root differing from 1 by less than ε. Define $W_\varepsilon \subset \overline{\mathbb{C}}$ as the set of complex number of modulus less than ε.

Lemma 17. *For $\varepsilon < \frac{1}{2}$ there exists an homeomorphism between the direct product of $V^n(\varepsilon)$ and $W(\varepsilon)$ and $U^{n+1}(\varepsilon)$. Moreover for any Young diagram Y this homeomorphism sends the stratum $(\Sigma^n Y \cap V^n(\varepsilon)) \times W(\varepsilon)$ to $\Sigma^{n+1} Y \cap U^{n+1}(\varepsilon)$.*

Proof. Consider the map $V^n(\varepsilon) \times W(\varepsilon) \to U^{n+1}(\varepsilon)$ sending $(p(z), y)$ to $p(z) \times (z - 1 + y)$. This map is continuous and onto. Since any polynomial in $U^{n+1}(\varepsilon)$ has only one root differing from 1 by less than ε, it is one-to-one. Moreover it is clear that this map respects the multiplicities of roots and hence for any Young diagram Y it sends $\Sigma^n Y$ to $\Sigma^{n+1} Y$. $\square$

Remark 18. The map defined in the proof of Lemma 17 is a local analog of the embedding Φ^n defined in Section 2.

Define ${}^*\overline{\Sigma}^n Y$ as the space $\overline{\Sigma}^n Y$ with the points $z^{n-1}(z-1)$ removed. The main results of this section is the following lemma:

Lemma 19. *There exists an exact sequence* $(i \geqslant 0)$

$$\to H_{(i)}\bigl({}^*\overline{\Sigma}^n Y\bigr) \to H_{(i)}\bigl(\overline{\Sigma}^n Y\bigr) \to H_{(i)}\bigl(\overline{\Sigma}^{n-1} Y\bigr) \to H_{(i+1)}\bigl({}^*\overline{\Sigma}^n Y\bigr) \to .$$

Proof. Consider the exact sequence of the pair $(\overline{\Sigma}^n Y, \overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon))$:

$$\begin{aligned}&\to H_{(i)}\bigl(\overline{\Sigma}^n Y/\overline{U}^n(\varepsilon)\bigr) \to H_{(i)}\bigl(\overline{\Sigma}^n Y\bigr) \to H_{(i)}\bigl(\overline{\Sigma}^n Y, \bigl(\overline{\Sigma}^n Y/\overline{U}^n(\varepsilon)\bigr)\bigr)\\ &\to H_{(i+1)}\bigl(\overline{\Sigma}^n Y/\overline{U}^n(\varepsilon)\bigr) \to .\end{aligned}$$

By Lemmas 15 and 17, $\overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon)$ is a deformation retract of ${}^*\overline{\Sigma}^n Y$. Hence the homology groups of $\overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon)$ and ${}^*\overline{\Sigma}^n Y$ are isomorphic. By Lemma 17 the space $\overline{\Sigma}^n Y \setminus (\overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon)) \approx \overline{U}^n(\varepsilon)$ is homeomorphic to the suspension $\overline{V}^{n-1}(\varepsilon) \sharp \overline{W(\varepsilon)}$. By Corollary 16 this implies that the homology groups of $(\overline{\Sigma}^n Y, (\overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon))$ and $(\overline{\Sigma}^{n-1} Y)$ are isomorphic. Hence the exact sequence of the lemma follows from the exact sequence of the pair $(\overline{\Sigma}^n Y, \overline{\Sigma}^n Y \setminus \overline{U}^n(\varepsilon))$. $\square$

Given a subspace X of P^n and a sphere $\overline{\mathbb{C}}^k$, define ${}^*(\overline{\mathbb{C}}^k \sharp X)$ as the suspension $\overline{\mathbb{C}}^k \sharp {}^*X$. The commutative diagram (7) gives rise to a commutative diagram with stars:

$$\begin{array}{ccccccc}
H_{(i-1)}({}^*A) & \longrightarrow & H_{(i+\mu-1)}({}^*S) & \longrightarrow & H_{(i+\mu-1)}({}^*S/{}^*A) & \longrightarrow & H_{(i)}({}^*A)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H_{(i-1)}({}^*B) & \longrightarrow & H_{(i+\mu-1)}({}^*\overline{\Sigma}^n Y) & \longrightarrow & H_{(i+\mu-1)}({}^*\overline{\Sigma}^n Y/{}^*B) & \longrightarrow & H_{(i)}({}^*B)
\end{array} \tag{8}$$

5. Proofs of the main theorems

Consider the following three assertions where q and s are positive integers and $s < \gcd Y$:

$$\begin{aligned}
&A(Y,q)\colon\ H_{(i)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y + q \gcd Y + s} Y\bigr) = 0 \quad \text{for all } i \text{ if } s \neq 0,\\
&B(Y,q)\colon\ H_{(i)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y + (q+1)\gcd Y} Y\bigr) \quad \text{is a finite group for all } i,\\
&C(Y,q)\colon\ H_{(i)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y + (q+1)\min Y} Y\bigr) = 0 \quad \text{for } i \leqslant q(2\min Y - 3) + \operatorname{codim} Y - 1,
\end{aligned}$$

Consider the spaces S, A and B defined at the end of Section 3. By Eq. (6) the homology of these spaces coincides with the homology of $\overline{\Sigma}^{n-n_k m_k}(Y')$, $\overline{\Sigma}^{n-n_k m_k}(Y'')$ and $\overline{\Sigma}^{n}(Y''')$. Moreover these isomorphisms also occur when stars are placed on all spaces:

$$\begin{aligned}
&H_{(i)}({}^*S) = H_{(i)}\bigl({}^*\overline{\Sigma}^{n-n_k m_k}(Y')\bigr),\\
&H_{(i)}({}^*A) = H_{(i)}\bigl({}^*\overline{\Sigma}^{n-n_k m_k}(Y'')\bigr),\\
&H_{(i)}({}^*B) = H_{(i)}\bigl({}^*\overline{\Sigma}^{n}(Y''')\bigr).
\end{aligned}$$

According to these equations, in the commutative diagram (8) we can replace ${}^*S$ by ${}^*\overline{\Sigma}^{n-n_k m_k}(Y')$, ${}^*A$ by ${}^*\overline{\Sigma}^{n-n_k m_k}(Y'')$ and ${}^*B$ by ${}^*\overline{\Sigma}^{n}(Y''')$.

The proof of any assertion A, B, C can be reduced to the proof that some homology groups of ${}^*\overline{\Sigma}^{n}Y$, with coefficients in $\mathbb{Z}$ or $\mathbb{C}$, are zero. In each case the proof will be by induction on q and on the number of distinct integers in the Young diagram Y, using the commutative diagram (8) where the spaces ${}^*A$, ${}^*S$ and ${}^*B$ are replaced by the corresponding Young strata as explained above. The scheme of the proof is the same in each case. Consider for instance assertion A (in this case we consider homology groups with coefficients in $\mathbb{Z}$). Suppose that $A(Y,0)$ is true for all Y and that $A(Y,q)$ is true for all void Y (of course such an assertion is trivial). Consider an arbitrary Young diagram Y and three integers q, n and s such that $n = \operatorname{sum} Y + q \gcd Y + s$, with $0 < s < \gcd Y$. To prove assertion $A(Y,q)$ we can suppose by induction that the group $H_{(i+\mu-1)}({}^*\overline{\Sigma}^{n-n_k m_k}(Y'))$ is zero because there are less distinct integers in the Young diagram Y' than in Y. Moreover considering the integers $q(Y'')$ and $q(Y''')$ defined by the equations $n - n_k m_k = \operatorname{sum}(Y'') + q(Y'')\gcd(Y'') + s$ and $n = \operatorname{sum}(Y''') + q(Y''')\gcd(Y''') + s$ it is easy to check that $q(Y'') < q$ and $q(Y''') < q$ ($\gcd(Y'') = \gcd(Y''') = \gcd Y$, $\operatorname{sum}(Y'') = \operatorname{sum} Y - n_k m_k + m_k$ and $\operatorname{sum}(Y''') = \operatorname{sum} Y + m_k$). Hence by induction we can suppose that the groups $H_{(i-1)}({}^*\overline{\Sigma}^{n-n_k m_k}(Y''))$ and $H_{(i-1)}({}^*\overline{\Sigma}^{n}(Y'''))$ are zero. So according to the induction hypothesis, the commutative diagram (8) degenerates into:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H_{(i+\mu-1)}({}^*S/{}^*A) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H_{(i+\mu-1)}({}^*\overline{\Sigma}^{n}Y) \arrow[r] & H_{(i+\mu-1)}({}^*\overline{\Sigma}^{n}Y/{}^*B) \arrow[r] & 0
\end{tikzcd} \tag{9}$$

By Lemma 12 the vertical arrow between the homology groups $H_{(i)}({}^*S/{}^*A)$ and $H_{(i)}({}^*\overline{\Sigma}^{n}Y/{}^*B)$ is an isomorphism. Hence the group $H_{(i+\mu-1)}({}^*\overline{\Sigma}^{n}Y)$ is zero. By induction, if $A(Y,0)$ is true for all Young diagrams Y then $A(Y,q)$ is true for all Y and all positive integer q.

Lemma 20. *For any integer s such that* $0 \leqslant s < \min Y$:

$$\begin{aligned}
&H_{(\operatorname{codim} Y-1)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y}Y\bigr) = \mathbb{Z},\\
&H_{(\operatorname{codim} Y-1)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y+s}Y\bigr) = 0 \quad \text{if } s \neq 0,\\
&H_{(i)}\bigl({}^*\overline{\Sigma}^{\operatorname{sum} Y+s}Y\bigr) = 0 \quad \text{for } i \neq \operatorname{codim} Y - 1.
\end{aligned}$$

Since $\min Y \geqslant \gcd Y$, this lemma implies assertions $A(Y,0)$ for all Young diagrams Y. In particular according to the preceding assertions, this lemma implies assertion $A(Y,q)$ for all Y and $q \geqslant 0$.

Proof. By Lemma 14, the space $\overline{\Sigma}^{\operatorname{sum} Y+s} Y$ is isomorphic to a sphere. Moreover if $s \neq 0$ then $z^{\operatorname{sum} Y+s-1}(z-1)$ belongs to $\overline{\Sigma}^{\operatorname{sum} Y+s} Y$. Hence the space ${}^*\overline{\Sigma}^{\operatorname{sum} Y+s} Y$ is contractible and all its reduced homology groups are zero. On the contrary if $s=0$ then $z^{\operatorname{sum} Y+s-1}(z-1)$ does not belong to $\overline{\Sigma}^{\operatorname{sum} Y+s} Y$ and ${}^*\overline{\Sigma}^{\operatorname{sum} Y+s} Y$ is a sphere. Hence $H_{(0)}({}^*\overline{\Sigma}^{\operatorname{sum} Y} Y)=\mathbb{Z}$ and all other reduced homology groups are zero. $\square$

Lemma 21 [1]. *For any Young diagram $(m_1,\ldots,m_1)$ consisting of n_1 times the integer m_1, the homology groups of the space ${}^*\overline{\Sigma}^{(n_1+1)m_1}(m_1,\ldots,m_1)$ are given by*:

$$H_{(2m_1-2)}\big({}^*\overline{\Sigma}^{(n_1+1)m_1}(m_1,\ldots,m_1)\big)=\mathbb{Z}_{n_1+1},$$
$$H_{(i)}\big({}^*\overline{\Sigma}^{(n_1+1)m_1}(m_1,\ldots,m_1)\big)=0 \quad \textit{for } i\neq 2m_1-2.$$

This lemma is formula (18) of [1]. It implies assertion $B(Y,0)$ and $C(Y,0)$ for all Y consisting of a collection of identical integers. Moreover the assertions $B(Y,q)$ and $C(Y,q)$ are trivial for any $q \geqslant 0$ when Y is void.

Remark 22. The integer n_1+1 in the lemma is the number of branches of ΣY intersecting along a generic point of its self intersection stratum.

The idea of the proof is as follows (see [1]). Given the condition of the lemma, the space S is a sphere. Hence all homology groups $H_{(i)}(S)$ with $i>0$ are zero and $H_{(0)}(S)=\mathbb{Z}$. In particular, the sequence with stars given in the end of Section 4 gives rise to a short exact sequence:

$$\mathbb{Z}\to\mathbb{Z}\to H_{(2m_1-2)}\big({}^*\overline{\Sigma}^{(n_1+1)m_1} Y\big)\to\mathbb{Z}\to\mathbb{Z},$$

where the first and last arrows are induced by maps of degree n_1+1 [1, Lemma 1]. Hence:

$$H_{(2m_1-2)}\big({}^*\overline{\Sigma}^{(n_1+1)m_1}(m_1,\ldots,m_1)\big)=\mathbb{Z}_{n_1+1}.$$

The same argument implies that all other reduced homology groups are zero. $\square$

Using this lemma the proof of assertion $B(Y,q)$ by induction is the same as the proof of assertion $A(Y,q)$ given above, considering homology groups with coefficients in $\mathbb{C}$. To prove assertion $C(Y,q)$ it is sufficient to notice in the induction scheme that $\min Y$ is equal to $\min(Y'')$ and $\min(Y''')$.

Proof of repetition theorem. Assertions $A(Y,q)$ and the exact sequence of Lemma 19 imply:

$$H_{(i)}\big(\overline{\Sigma}^{\operatorname{sum} Y+q\gcd Y+s} Y\big)=H_{(i)}\big(\overline{\Sigma}^{\operatorname{sum} Y+q\gcd Y} Y\big),$$

for all i and s such that $0\leqslant s<\gcd Y$. The theorem follows by Alexander duality. $\square$

Proof of finiteness theorem. Assertions $B(Y,q)$ and the exact sequence of Lemma 19 imply that for any integer $n>\operatorname{sum} Y+\gcd Y-1$ the following isomorphism of homology groups with coefficients in $\mathbb{C}$ holds:

$$H_{(i)}\big(\overline{\Sigma}^{n} Y,\mathbb{C}\big)=H_{(i)}\big(\overline{\Sigma}^{\operatorname{sum} Y+\gcd Y-1} Y,\mathbb{C}\big).$$

Moreover according to Lemma 20, for $0 \leqslant s \leqslant \gcd Y - 1$:

$$H_{(i)}\big(\overline{\Sigma}^{\operatorname{sum} Y+s} Y, \mathbb{C}\big) = H_{(i)}(\overline{\Sigma}^{\operatorname{sum} Y} Y, \mathbb{C}).$$

For $i \neq \operatorname{codim} Y - 1$, $H_{(i)}(\overline{\Sigma}^{\operatorname{sum} Y} Y, \mathbb{C}) = 0$. Moreover $H_{(\operatorname{codim} Y-1)}(\overline{\Sigma}^{\operatorname{sum} Y} Y, \mathbb{C}) = \mathbb{C}$. The theorem follows using Alexander duality. □

Proof of stabilization theorem. This follows from the exact sequence of Lemma 19 and assertion $C(Y, q)$ as in the case of the repetition theorem. □

6. Tables

In this section we give the results of explicit computations of cohomology groups in lowest dimensions. These computations are based on cellular decompositions of the spaces P^n compatible with the Young stratification (each stratum is a union of cells). These cellular decompositions are described in [7]. These tables, together with the results of [1], give the complete description of cohomology groups of strata of complex codimension less than 5 in spaces of polynomials of degree less than 10. In particular, for the sake of completeness, we have included the results for the strata $\Sigma(3)$, $\Sigma(4)$ and $\Sigma(5)$ although they do not correspond to multisingularities. In each case the stable groups, according to the stabilisation theorem above, are underlined.

Table 1
$H^i(P^n \setminus \Sigma(3))$

	$i=0$	$i=3$	4	5	6	7	8	9	10	11
$n=3,4,5$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$						
$n=6,7,8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\underline{0}$			
$n=9,10,11$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\underline{0}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\underline{\mathbb{Z}_3}$

Table 2
$H^i(P^n \setminus \Sigma(3,2))$

	$i=0$	$i=5$	6	7	8	9	10	11
$n=5$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$						
$n=6$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	0	$\mathbb{Z}_2$				
$n=7$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2^2$				
$n=8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2^2$				
$n=9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{\mathbb{Z}_2^2}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3$

Table 3
$H^i(P^n \setminus \Sigma(4))$

	$i=0$	$i=5$	6	7	8	9	10	11	12	13
$n=4,5,6,7$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$	$\underline{0}$					
$n=8,9,10,11$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\underline{0}$	$\underline{0}$

Table 4
$H^i(P^n \setminus \Sigma(3,3))$

	$i=0$	$i=7$	8	9	10	11	12
$n=6,7,8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$			
$n=9,10,11$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\mathbb{Z}_3}$	$\underline{0}$

Table 5
$H^i(P^n \setminus \Sigma(4,2))$

	$i=0$	$i=7$	8	9	10	11
$n=6,7$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$				
$n=8,9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$n=10,11$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}_6$

Table 6
$H^i(P^n \setminus \Sigma(5))$. For $5 \leqslant n \leqslant 9$ the first twelve groups are stable, for $10 \leqslant n \leqslant 14$, the first eighteen groups are stable

	$i=0$	$i=7$	$i=15$
$5 \leqslant n \leqslant 9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	
$10 \leqslant n \leqslant 14$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}_2}$

Table 7
$H^i(P^n \setminus \Sigma(4,3))$

	$i=0$	$i=9$	10	11	12	13
$n=7$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$				
$n=8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2$		
$n=9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$		
$n=10$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\underline{\mathbb{Z}_2}$	$\underline{0}$	$\mathbb{Z}_2$

Table 8
$H^i(P^n \setminus \Sigma(5,2))$

	$i=0$	$i=9$	10	11	12	13	14	15
$n=7$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$						
$n=8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$						
$n=9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2$				
$n=10$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$

Table 9
$H^i(P^n \setminus \Sigma(3,3,2))$

	$i=0$	$i=9$	10	11
$n=8$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$		
$n=9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	0	$\mathbb{Z}_3$
$n=10$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_6$

Table 10
$H^i(P^n \setminus \Sigma(4,2,2))$

	$i=0$	$i=9$	10	11
$n=8,9$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	0	$\mathbb{Z}_2$
$n=10,11$	$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}$	$\underline{0}$	$\mathbb{Z}_6$

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